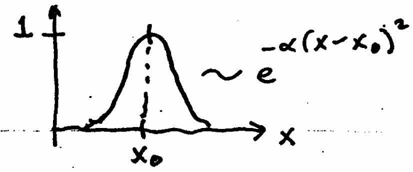
Gaussian Integrals:



① $$I = \int_{-\infty}^{+\infty} dx\, e^{-\alpha(x-x_0)^2}$$

$y = x - x_0$

$dy = dx$

$$= \int_{-\infty}^{+\infty} dy\, e^{-\alpha y^2}$$

$\therefore$ Location of x_0 does not change area under curve as long as limits are $-\infty \to +\infty$.

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② Evaluating I: $\quad I = I(\alpha)$

$$I^2 = \left(\int_{-\infty}^{+\infty} dx\, e^{-\alpha x^2}\right)\left(\int_{-\infty}^{+\infty} dy\, e^{-\alpha y^2}\right)$$

$$= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, e^{-\alpha(x^2+y^2)}$$

Use "plane polar coordinates"

$r^2 = x^2 + y^2$

$\tan\phi = y/x$

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy = \int_0^{\infty} dr\, r \int_0^{2\pi} d\phi$$

[Note: "Jacobian factor"

$$= \int_0^{\infty} dr\, r \underbrace{\int_0^{2\pi} d\phi}\, e^{-\alpha r^2}$$

$$= (2\pi) \int_0^{\infty} \frac{dy}{2\alpha}\, e^{-y}$$

$y = \alpha r^2$

$dy = 2\alpha\, r\, dr$

$r\,dr = dy/2\alpha$

$$= \left(\frac{\pi}{\alpha}\right) \int_0^{\infty} dy\, e^{-y}$$

$$= \left(\frac{\pi}{\alpha}\right) \frac{e^{-y}}{-1}\Big|_0^{\infty} = \left(\frac{\pi}{\alpha}\right)(1-0) = \frac{\pi}{\alpha}$$

Thus $I^2 = \frac{\pi}{\alpha}$ so $\boxed{I = \sqrt{\frac{\pi}{\alpha}} = \int_{-\infty}^{+\infty} dx\, e^{-\alpha(x-x_0)^2}}$

③ <u>Evaluating "moments" of the distubutem:</u>

E.g. $\int_{-\infty}^{+\infty} dx\, x^2\, e^{-\alpha(x-x_0)^2}$

$y = x - x_0$
$dy = dx$
$x = y + x_0$

$$= \int_{-\infty}^{+\infty} dy\, (y+x_0)^2\, e^{-\alpha y^2}$$

$$= \int_{-\infty}^{+\infty} dy\, (y^2 + 2x_0 y + x_0^2)\, e^{-\alpha y^2}$$

$$= \underbrace{\int_{-\infty}^{+\infty} dy\, y^2\, e^{-\alpha y^2}}_{?} + \underbrace{2x_0 \int_{-\infty}^{+\infty} dy\, y\, e^{-\alpha y^2}}_{0} + \underbrace{x_0^2 \int_{-\infty}^{+\infty} dy\, e^{-\alpha y^2}}_{x_0^2\sqrt{\frac{\pi}{\alpha}}}$$

(Called the "second moment")　　(why?)

<u>Trick:</u> $x e^{xy} \overset{?}{=} \left(\frac{\partial}{\partial y} e^{xy}\right)_x = e^{xy} \left(\frac{\partial xy}{\partial y}\right)_x = x e^{xy}$ ✓

$$\therefore \boxed{y^2 e^{-\alpha y^2} = \frac{\partial}{\partial(-\alpha)} e^{-\alpha y^2}}$$

Thus $\int_{-\infty}^{+\infty} dy\, y^2 e^{-\alpha y^2} = \int_{-\infty}^{+\infty} dy \left(\frac{\partial}{\partial(-\alpha)}\right) e^{-\alpha y^2}$

$$= \frac{\partial}{\partial(-\alpha)} \int_{-\infty}^{+\infty} dy\, e^{-\alpha y^2} = \frac{\partial}{\partial(-\alpha)} \left(\frac{\pi}{\alpha}\right)^{1/2}$$

$$= -\sqrt{\pi}\, \frac{\partial}{\partial\alpha} \alpha^{-1/2} = \left(-\sqrt{\pi}\right)\left(-\frac{1}{2}\right) \alpha^{-3/2}$$

$$\boxed{= \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\, \frac{1}{\alpha} = \int_{-\infty}^{+\infty} dy\, y^2 e^{-\alpha y^2}}$$

Using this method we get:

$$\int_{-\infty}^{+\infty} dy\, y^{2n} e^{-\alpha y^2} = \sqrt{\frac{\pi}{\alpha}} \cdot \frac{1}{(2\alpha)^n} \left[1 \times 3 \times 5 \times \cdots \times (2n-1)\right]$$

= 2x integral on McQuarrie back cover

$$\int_{-\infty}^{+\infty} dy\, y^{2n+1} e^{-\alpha y^2} = 0 \quad \text{by symmetry}$$

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